

# Corrections and Addendum to “Inverse Spectral Analysis with Partial Information on the Potential, III. Updating Boundary Conditions”

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Unless explicitly stated otherwise, all subsequent formula numbers, references, cited theorems, page numbers, etc., refer to those in [1].

(1) Our proof of Lemma 2.2 contains a number of typographical errors:

(a)  $\sinh(2|y|^{1/2})$  should be replaced by  $\sinh(|2y|^{1/2})$  in (2.3). Similarly,  $\exp(2|y|^{1/2})$  should be replaced by  $\exp(|2y|^{1/2})$  in the 2d and 4th lines on p. 755.

(b) The infinite product in (2.4) should read

$$\prod_{n=1}^{\infty} \left[ \left( 1 + \frac{16|y|^2}{\pi^4 n^4} \right)^{1/2} \left( 1 + \frac{|y|^2}{\lambda_n^2} \right)^{-1/2} \right],$$

and the argument following (2.4) should be replaced by the following one.

If  $0 \leq a \leq b$ , then

$$\frac{1 + a^2|y|^2}{1 + b^2|y|^2} \leq 1;$$

and if  $a > b > 0$ , then

$$\begin{aligned} \left( \frac{1 + a^2|y|^2}{1 + b^2|y|^2} \right)^{1/2} &= \left( 1 + \frac{(a^2 - b^2)|y|^2}{1 + b^2|y|^2} \right)^{1/2} \leq \left( 1 + \frac{a^2 - b^2}{b^2} \right)^{1/2} = \frac{a}{b}, \\ \prod_{n=1}^{\infty} \left( \frac{1 + \frac{16|y|^2}{\pi^4 n^4}}{1 + \frac{|y|^2}{\lambda_n^2}} \right)^{1/2} &\leq \prod_{n: \lambda_n > \pi^2 n^2 / 4}^{\infty} \frac{4\lambda_n}{\pi^2 n^2} = \prod_{n=1}^{\infty} \left( 1 + \frac{(\lambda_n - (\pi^2 n^2 / 4))_+}{\pi^2 n^2 / 4} \right) < \infty \end{aligned}$$

if (2.1) holds.

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(c) Replace the estimates (2.5a) and (2.5b) by the asymptotic relations

$$P_j(z) \underset{z \rightarrow \pm i\infty}{=} O(\cos(\sqrt{z})) \quad \text{and} \quad Q_j(z) \underset{z \rightarrow \pm i\infty}{=} O(\sqrt{z} \sin(\sqrt{z})),$$

respectively.

(d) Replace  $f(iy)$  by  $|f(iy)|$  in the 4th line of p. 755.

(2) A slight modification of our proof of Theorem 2.1 allows us to also recover Marchenko's uniqueness result (Theorem 2.3.2 in [16]), which goes beyond Borg's theorem (our Corollary 3.2) in the following sense. Marchenko does not a priori fix the boundary conditions when comparing spectra for the two potentials  $q_1$  and  $q_2$ . More precisely, denote by  $\sigma(q; h, \ell)$  the spectrum of the operator  $-d^2/dx^2 + q$  in  $L^2((0, 1))$  with boundary conditions  $u'(0) + hu(0) = 0$ ,  $u'(1) + \ell u(1) = 0$ ,  $h, \ell \in \mathbb{R}$ . Then Marchenko's result reads as follows.

**Theorem** (Theorem 2.3.2 in [16]). Let  $q_1, q_2 \in L^1((0, 1))$  be real-valued, suppose that  $h_j, k_j, \ell_j \in \mathbb{R}$ ,  $j = 1, 2$ ,  $h_1 \neq k_1$ ,  $h_2 \neq k_2$ , and assume

$$\sigma(q_1; h_1, \ell_1) = \sigma(q_2; h_2, \ell_2), \quad \sigma(q_1; k_1, \ell_1) = \sigma(q_2; k_2, \ell_2). \quad (1)$$

Then  $h_1 = h_2$ ,  $k_1 = k_2$ ,  $\ell_1 = \ell_2$ , and  $q_1 = q_2$  a.e. on  $[0, 1]$ .  $\square$

*Sketch of proof.* The asymptotic eigenvalue behavior (3.1) yields

$$h_1 - h_2 = k_1 - k_2 := A.$$

Identifying  $m_1(z) = m_{\ell_1}(z)$  and  $m_2(z) = m_{\ell_2}(z)$ , and redefining  $H(z)$  as

$$\tilde{H}(z) = P_2(z)Q_1(z) - P_1(z)Q_2(z) + AP_1(z)P_2(z)$$

and  $G(z)$  as

$$\tilde{G}(z) = \tilde{H}(z)/f(z) = P_1(z)P_2(z)[m_1(z) - m_2(z) + A]/f(z),$$

one can follow the proof of Theorem 2.1 step by step. As a result, one arrives at

$$|\tilde{G}(iy)| \leq \frac{\exp(|2y|^{1/2})}{|f(iy)|} |m_1(iy) - m_2(iy) + A| \underset{|y| \rightarrow \infty}{=} O(|y|^{-1/2}) \quad (2)$$

and hence at  $\tilde{G}(z) = 0$  using a Phragmén-Lindelöf argument. Thus,

$$m_1(z) - m_2(z) + A = 0,$$

and the asymptotic behavior

$$m_j(z) \underset{z \rightarrow \pm i\infty}{=} \pm i\sqrt{z} + o(1)$$

then proves  $\Lambda = 0$ . Hence,

$$m_1(z) = m_2(z) \text{ and } h_1 = h_2, \quad k_1 = k_2.$$

But then  $m_{\ell_1}(z) = m_{\ell_2}(z)$  yields  $\ell_1 = \ell_2$  and  $q_1 = q_2$  a.e. on  $[0, 1]$  by Marchenko's Theorem 1.1. ■

We emphasize the  $O(|y|^{-1/2})$ -term in (2), as opposed to the corresponding  $o(|y|^{-1/2})$ -term in our proof of Theorem 2.1. This shows that in contrast to Borg's theorem (i.e., Corollary 3.2), all eigenvalues are required in (1), and one can no longer dispense with the knowledge of one of them.

Actually, Marchenko also includes the case of Dirichlet boundary conditions by reducing it to the case discussed above by means of appropriate linear fractional transformations of  $m(z)$ . We omit further details.

### Acknowledgments

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### References

- [1] Rafael del Rio, Fritz Gesztesy, and Barry Simon, *Inverse spectral analysis with partial information on the potential, III. Updating boundary conditions*, Internat. Math. Res. Notices (IMRN) **1997**, 751–758.

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